

GAUSS-MARKOV PROCESSES AS SPACE-TIME SCALED STATIONARY ORNSTEIN-UHLENBECK PROCESSES

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ABSTRACT. We present a class of Gauss-Markov processes which can be represented as space-time scaled stationary Ornstein-Uhlenbeck processes defined on the real line. To give examples, we study scaled Wiener bridges, Ornstein-Uhlenbeck type bridges, weighted Wiener bridges and so called F -Wiener bridges. By giving counterexamples, we also point out that this kind of representation does not hold in general, e.g., for a zero area Wiener bridge. To give a possible application, we show that our results can be useful to calculate the distribution of the supremum location of certain standardized Gauss-Markov processes on compact time intervals.

1. INTRODUCTION

In this paper, we present a class of Gauss-Markov processes which can be represented as space-time scaled stationary Ornstein-Uhlenbeck processes defined on the real line by specifying the space and time transformations in question explicitly as well. To motivate our method, we will first present the well-known example that a Wiener bridge can be represented as a space-time scaled stationary Ornstein-Uhlenbeck process, see, e.g., Shorack and Wellner [22, Exercise 10, page 32]. Let $(B_t)_{t \geq 0}$ be a standard Wiener process, then its Lamperti transform (see Lamperti [15, page 64])

$$(1.1) \quad S_t := e^{-\frac{t}{2}} B_{e^t}, \quad t \in \mathbb{R},$$

defines a strictly stationary centered Gauss process $S = (S_t)_{t \in \mathbb{R}}$ defined on the real line with

$$(1.2) \quad \text{Cov}(S_s, S_t) = e^{-\frac{|t-s|}{2}}, \quad s, t \in \mathbb{R},$$

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see, e.g., Doob [10] or Shorack and Wellner [22, Exercise 9, page 32]. The process S is known as a stationary Ornstein-Uhlenbeck process defined on \mathbb{R} . Then a Wiener bridge from 0 to 0 over the time interval $[0, 1]$ generates the same law on $C([0, 1])$ as the space-time scaled stationary Ornstein-Uhlenbeck process

$$U_t := \begin{cases} \sqrt{t(1-t)} S\left(\ln\left(\frac{t}{1-t}\right)\right) & \text{if } t \in (0, 1), \\ 0 & \text{if } t = 0 \text{ or } t = 1, \end{cases}$$

see, e.g., Shorack and Wellner [22, Exercise 10, page 32], where for a subset $D \subset [0, \infty)$, $C(D)$ denotes the space of continuous real-valued functions defined on D . Recall that the law of the pathwise unique strong solution of the stochastic differential equation (SDE)

$$dZ_t = -\frac{1}{1-t} Z_t dt + dB_t, \quad t \in [0, 1),$$

with an initial value $Z_0 = 0$ coincides with that of the Wiener bridge from 0 to 0 over the time interval $[0, 1]$.

As a generalization of the observation above, in Section 2 we provide a class of Gauss-Markov processes (satisfying a linear SDE) which can be represented as space-time scaled stationary Ornstein-Uhlenbeck processes defined on the real line by specifying the space and time transformations in question explicitly, see Theorem 2.1. We also formulate two consequences of Theorem 2.1, see Proposition 2.4 and Theorem 2.5 (the second one covers the case that the underlying Gauss-Markov process is a bridge). In Remark 2.3 we compare our results with the corresponding ones of Lachout [14] who investigated a related problem. In Section 3, we give some examples: scaled Wiener bridges (also called general α -Wiener bridges), Ornstein-Uhlenbeck type bridges, weighted Wiener bridges and so called F -Wiener bridges. In Section 4, we present counterexamples where the representation in question does not hold such as the zero area Wiener bridge. To give a possible application of our results, we point out that our main Theorem 2.1 enables us to calculate the distribution of the supremum location of certain standardized Gauss-Markov processes on compact time intervals, see Section 5.

2. A GENERAL FRAMEWORK

In what follows, let \mathbb{R}_+ denote the set of non-negative real numbers. For $s, t \in \mathbb{R}$, let $s \wedge t$ denote $\min(s, t)$, and let $\mathcal{B}(\mathbb{R})$ denote the set of Borel sets of \mathbb{R} . Recall that $C([0, T])$ with $T \in (0, \infty)$, and $C([0, \infty))$ are complete, separable metric spaces (with appropriate metrics). Due to the strictly increasing and continuous time change $\frac{2T}{\pi} \arctan s$, $s \in [0, \infty)$ (which is a bijection between $[0, \infty)$ and $[0, T)$), we get $C([0, T])$ is a complete, separable metric space as well.

Let $T \in [0, \infty]$. Let $\phi : [0, T) \rightarrow (0, \infty)$ be a continuously differentiable function with $\phi(0) = 1$, $\psi, \sigma : [0, T) \rightarrow \mathbb{R}$ be continuous functions, and suppose that $\sigma(t) \neq 0$ on some interval $(0, \delta)$ for some $\delta \in (0, T]$. Let us consider the SDE

$$(2.1) \quad dZ_t = \left(\frac{\phi'(t)}{\phi(t)} Z_t + \psi(t) \right) dt + \sigma(t) dB_t, \quad t \in [0, T),$$

with a non-random initial value $Z_0 = \xi \in \mathbb{R}$, where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ satisfying the usual conditions. Note that in the drift coefficient of the SDE (2.1) the factor $\phi'(t)/\phi(t)$ can be an arbitrary continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, since the Cauchy problem $\frac{\phi'(t)}{\phi(t)} = f(t)$, $t \in \mathbb{R}_+$, with $\phi(0) = 1$ has the unique solution

$$\phi(t) = \exp \left\{ \int_0^t f(u) du \right\}, \quad t \in \mathbb{R}_+.$$

However, we keep the form $\phi'(t)/\phi(t)$ in order to have a more compact presentation (especially, for the solution of the SDE (2.1) given in (2.2)) and also for the examples presented in Section 3 later on.

Since ϕ'/ϕ , ψ and σ are non-random, measurable and locally bounded, by using Itô's formula,

$$(2.2) \quad Z_t = \phi(t) \left(\xi + \int_0^t \frac{\psi(u)}{\phi(u)} du + \int_0^t \frac{\sigma(u)}{\phi(u)} dB_u \right), \quad t \in [0, T),$$

can be shown to be the pathwise unique strong solution of the SDE (2.2). The Gauss-Markov process Z is called a process of Ornstein-Uhlenbeck type with parameters ϕ , ψ and σ in Patie [18, pages 49 and 58]. We also note that processes of the form (2.2) have recently found an application in Alili and Patie [2, second proof of Theorem 1.2].

One can easily calculate

$$\text{Cov}(Z_s, Z_t) = \phi(s)\phi(t) \int_0^{s \wedge t} \frac{\sigma(u)^2}{\phi(u)^2} du, \quad s, t \in [0, T].$$

Let us consider the mean centered process defined by

$$(2.3) \quad \tilde{Z}_t := Z_t - \mathbb{E}(Z_t) = \phi(t) \int_0^t \frac{\sigma(u)}{\phi(u)} dB_u, \quad t \in [0, T].$$

Theorem 2.1. *There exists a strictly stationary centered Ornstein-Uhlenbeck process $R = (R_t)_{t \in \mathbb{R}}$ with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$, such that*

$$(2.4) \quad \tilde{Z}_t = \phi(t) \sqrt{\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du} R \left(\ln \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) \right), \quad \forall t \in [0, T] \quad a.s.,$$

where \tilde{Z} is defined in (2.3), and the right hand side of (2.4) for $t = 0$ is understood as an almost sure limit as $t \downarrow 0$. Roughly speaking, the mean centered process $(\tilde{Z}_t)_{t \in [0, T]}$ coincides almost surely with a space-time scaled stationary Ornstein-Uhlenbeck process.

Proof. The proof consists of two parts: first we check that the pathwise unique strong solution of the SDE (2.1) can be represented as a space-time transformed standard Wiener process, and then we use Lamperti transformation recalled in the introduction. Namely, by Dambis, Dubins and Schwarz lemma (see, e.g., Revuz and Yor [20, Chapter V, Theorems 1.6 and 1.7] or Karatzas and Shreve [12, Theorem 3.4.6 and Problem 3.4.7]), there exists a standard Wiener process $(W_t)_{t \in \mathbb{R}_+}$ (possibly on an enlargement of the original filtered probability space and stopped at $\lim_{t \uparrow T} \int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du$) such that

$$(2.5) \quad \tilde{Z}_t = \phi(t) \int_0^t \frac{\sigma(u)}{\phi(u)} dB_u = \phi(t) W \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right), \quad \forall t \in [0, T] \quad a.s.$$

Indeed, $\int_0^t \frac{\sigma(u)}{\phi(u)} dB_u$, $t \in [0, T]$, is a continuous L^2 -martingale, since $\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du < \infty$ for all $t \in [0, T]$, and we note that even if $\lim_{t \uparrow T} \int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du = \infty$ does not hold, one can apply Dambis, Dubins and Schwarz lemma. Let

$$R_t := e^{-\frac{t}{2}} W_{e^t}, \quad t \in \mathbb{R}.$$

Then, as it was recalled in the Introduction, R is a strictly stationary centered Ornstein-Uhlenbeck process with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$, and, by (2.5),

$$\phi(t) \sqrt{\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du} R \left(\ln \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) \right) = \phi(t) W \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) = \tilde{Z}_t$$

for all $t \in (0, T)$ almost surely. Further,

$$\begin{aligned} (2.6) \quad & \lim_{t \downarrow 0} \phi(t) \sqrt{\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du} R \left(\ln \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) \right) \\ &= \lim_{t \downarrow 0} \phi(t) W \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) = \phi(0) W_0 = 0 \quad \text{a.s.}, \end{aligned}$$

yielding (2.4). \square

Remark 2.2. Under the conditions of Theorem 2.1, if we additionally suppose that $\lim_{t \uparrow T} \phi(t) \sqrt{\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du} = 0$, then, since R is strictly stationary, by Slutsky's lemma, we have

$$\phi(t) \sqrt{\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du} R \left(\ln \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) \right)$$

converges in probability to 0 as $t \uparrow T$. Later, under some stronger additional assumptions, we will strengthen this statement, namely, instead of convergence in probability we will prove almost sure convergence, see Theorem 2.5. \square

Remark 2.3. Lachout [14] investigated a related problem. Namely, given a collection of stochastic integrals of non-random real functions with respect to a standard Wiener process, i.e.

$$\int_0^\infty a_\theta(u) dB_u, \quad \theta \in \Theta,$$

where $\Theta \subseteq \mathbb{R}$ is a non-empty set, $a_\theta : \mathbb{R}_+ \rightarrow \mathbb{R}$ belongs to $L^2(\mathbb{R}_+)$, $\theta \in \Theta$, and $(B_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process. Let $f : \Theta \rightarrow \mathbb{R}$ be a Borel measurable function. Then there exists a stationary centered Ornstein-Uhlenbeck process $O = (O_t)_{t \in \mathbb{R}}$ with $\text{Cov}(O_s, O_t) = e^{-|t-s|}$, $s, t \in \mathbb{R}$, such that

$$(2.7) \quad \int_0^\infty a_\theta(u) dB_u = O(f(\theta)) \quad \text{a.s. for all } \theta \in \Theta$$

if and only if

$$(2.8) \quad \int_0^\infty a_{\theta_1}(u) a_{\theta_2}(u) du = e^{-|f(\theta_1) - f(\theta_2)|} \quad \text{for all } \theta_1, \theta_2 \in \Theta,$$

see Lachout [14, Theorem 4.1]. Next, we apply Lachout's result to our model. Namely, let $\Theta := (0, T)$, and for all $t \in (0, T)$, let $a_t : \mathbb{R}_+ \rightarrow \mathbb{R}$,

$$a_t(u) := \mathbf{1}_{[0,t]}(u) \frac{\sigma(u)}{\phi(u)} \left(\int_0^t \frac{\sigma(r)^2}{\phi(r)^2} dr \right)^{-1/2}, \quad u \in \mathbb{R}_+.$$

Then, for all $t \in (0, T)$, we have $a_t \in L^2(\mathbb{R}_+)$ and

$$\frac{\tilde{Z}_t}{\sqrt{\text{Var}(Z_t)}} = \frac{Z_t - \mathbb{E}(Z_t)}{\sqrt{\text{Var}(Z_t)}} = \left(\int_0^t \frac{\sigma(r)^2}{\phi(r)^2} dr \right)^{-1/2} \int_0^t \frac{\sigma(r)}{\phi(r)} dB_r = \int_0^\infty a_\theta(u) dB_u.$$

Further, if $t_1 \leq t_2$, $t_1, t_2 \in (0, T)$, then

$$\int_0^\infty a_{t_1}(u) a_{t_2}(u) du = \frac{\int_0^{t_1} \frac{\sigma(r)^2}{\phi(r)^2} dr}{\left(\int_0^{t_1} \frac{\sigma(r)^2}{\phi(r)^2} dr \right)^{1/2} \left(\int_0^{t_2} \frac{\sigma(r)^2}{\phi(r)^2} dr \right)^{1/2}} = \left(\frac{\int_0^{t_1} \frac{\sigma(r)^2}{\phi(r)^2} dr}{\int_0^{t_2} \frac{\sigma(r)^2}{\phi(r)^2} dr} \right)^{1/2}.$$

Let $f : (0, T) \rightarrow \mathbb{R}$,

$$f(t) := \frac{1}{2} \ln \left(\int_0^t \frac{\sigma(r)^2}{\phi(r)^2} dr \right), \quad t \in (0, T).$$

Then

$$e^{-|f(t_1)-f(t_2)|} = e^{f(t_1)-f(t_2)} = \left(\frac{\int_0^{t_1} \frac{\sigma(r)^2}{\phi(r)^2} dr}{\int_0^{t_2} \frac{\sigma(r)^2}{\phi(r)^2} dr} \right)^{1/2}, \quad t_1 \leq t_2, \quad t_1, t_2 \in (0, T).$$

Hence, by Theorem 4.1 in Lachout [14], there exists a strictly stationary centered Ornstein-Uhlenbeck process $O = (O_t)_{t \in \mathbb{R}}$ with $\text{Cov}(O_s, O_t) = e^{-|t-s|}$, $s, t \in \mathbb{R}$, such that

$$\frac{\tilde{Z}_t}{\sqrt{\text{Var}(Z_t)}} = O(f(t)) \quad \text{a.s. for all } t \in (0, T).$$

Hence

$$\tilde{Z}_t = \phi(t) \sqrt{\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du} O \left(\frac{1}{2} \ln \left(\int_0^t \frac{\sigma(r)^2}{\phi(r)^2} dr \right) \right) \quad \text{a.s. for all } t \in (0, T).$$

By choosing $R_t := O_{t/2}$, $t \in \mathbb{R}$, we have

$$\tilde{Z}_t = \phi(t) \sqrt{\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du} R \left(\ln \left(\int_0^t \frac{\sigma(r)^2}{\phi(r)^2} dr \right) \right) \quad \text{a.s. for all } t \in (0, T).$$

Note that, based on this, we are not in the position to be able to check that (2.6) holds.

Now let us compare our Theorem 2.1 with Theorem 4.1 in Lachout [14]. Lachout [14] has a more general setup than ours, however he can prove less. Namely, Equation (2.7) holds almost surely for all $\theta \in \Theta$. Our model is a submodel of Lachout's model, however, we can prove more. Namely, Equation (2.4) holds for all $t \in [0, T)$ almost surely. The reason for being able to prove more is that in our special setup continuous martingales appear and we can use Dambis, Dubins and Schwarz lemma. In Lachout's general setup no (local) martingales show up, so he cannot take advantage of Dambis, Dubins and Schwarz lemma. Hence our Theorem 2.1 cannot be considered as a consequence of Lachout's results. \square

Next we formulate two consequences of Theorem 2.1.

Proposition 2.4. *If $\lim_{t \uparrow T} \phi(t) = 0$ and there exists some $\varepsilon > 0$ such that the function*

$$\phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{\frac{1}{2} + \varepsilon}, \quad t \in [0, T),$$

is bounded, then for the mean centered process $(\tilde{Z}_t)_{t \in [0, T)}$ we have $\mathbb{P}(\tilde{Z}_T := \lim_{t \uparrow T} \tilde{Z}_t = 0) = 1$, and there exists a strictly stationary centered Ornstein-Uhlenbeck process $R = (R_t)_{t \in \mathbb{R}}$ with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$, such that

$$\tilde{Z}_t = \phi(t) \sqrt{\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du} R \left(\ln \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) \right), \quad \forall t \in [0, T] \quad \text{a.s.},$$

where the right hand side of the above equation for $t = 0$ and for $t = T$ is understood as an almost sure limit as $t \downarrow 0$ and $t \uparrow T$, respectively.

Proof. By Theorem 2.1, in order to prove the statement we additionally need to check that

$$\lim_{t \uparrow T} \phi(t) \sqrt{\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du} R \left(\ln \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) \right) = \lim_{t \uparrow T} \phi(t) W \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) = 0$$

almost surely, where $(W_t)_{t \in \mathbb{R}_+}$ is the standard Wiener process appearing in the proof of Theorem 2.1. If $\int_0^T \frac{\sigma(u)^2}{\phi(u)^2} du := \lim_{t \uparrow T} \int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \in \mathbb{R}_+$, then

$$\begin{aligned} \lim_{t \uparrow T} \phi(t) W \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) &= \left(\lim_{t \uparrow T} \phi(t) \right) W \left(\int_0^T \frac{\sigma(u)^2}{\phi(u)^2} du \right) \\ &= 0 \cdot W \left(\int_0^T \frac{\sigma(u)^2}{\phi(u)^2} du \right) = 0 \quad \text{a.s.} \end{aligned}$$

If $\int_0^T \frac{\sigma(u)^2}{\phi(u)^2} du = \infty$, then

$$\lim_{t \uparrow T} \phi(t) W \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) = \lim_{t \uparrow T} \phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{\frac{1}{2} + \varepsilon} \frac{W \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)}{\left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{\frac{1}{2} + \varepsilon}} = 0$$

almost surely, where we used that

$$\lim_{s \rightarrow \infty} \frac{W_s}{s^{\frac{1}{2} + \eta}} = 0 \quad \text{a.s. for all } \eta > 0,$$

which follows by the law of iterated logarithm for a standard Wiener process (see, e.g., Revuz and Yor [20, Chapter II, Corollary 1.12]). Indeed,

$$\frac{W_s}{s^{\frac{1}{2} + \eta}} = \frac{\sqrt{2s \ln(\ln(s))}}{s^{\frac{1}{2} + \eta}} \frac{W_s}{\sqrt{2s \ln(\ln(s))}} = \frac{\sqrt{2 \ln(\ln(s))}}{s^\eta} \frac{W_s}{\sqrt{2s \ln(\ln(s))}}, \quad s > e,$$

where, by the law of iterated logarithm for a standard Wiener process,

$$\left(\frac{W_s}{\sqrt{2s \ln(\ln(s))}} \right)_{s > e}$$

is bounded almost surely, and, by \mathcal{L} 'Hospital's rule,

$$\lim_{s \rightarrow \infty} \frac{\sqrt{2 \ln(\ln(s))}}{s^\eta} = \lim_{s \rightarrow \infty} \frac{1}{\sqrt{2} \eta s^\eta \ln(s) \sqrt{\ln(\ln(s))}} = 0.$$

□

The next theorem covers the case when the process given by the SDE (2.1) is a bridge over the time interval $[0, T]$.

Theorem 2.5. *If the process $(Z_t)_{t \in [0, T]}$ given by the SDE (2.1) satisfies $\mathbb{P}(Z_T := \lim_{t \uparrow T} Z_t = b) = 1$ with some $b \in \mathbb{R}$, and there exists some $\varepsilon > 0$ such that the function*

$$\phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{\frac{1}{2} + \varepsilon}, \quad t \in [0, T],$$

is bounded, then there exists a strictly stationary centered Ornstein-Uhlenbeck process

$R = (R_t)_{t \in \mathbb{R}}$ with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$, such that

$$\tilde{Z}_t = \phi(t) \sqrt{\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du} R \left(\ln \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) \right), \quad \forall t \in [0, T] \quad \text{a.s.},$$

where the right hand side of the above equation for $t = 0$ and for $t = T$ is understood as an almost sure limit as $t \downarrow 0$ and $t \uparrow T$, respectively.

Proof. Since $(Z_t)_{t \in [0, T]}$ is a Gauss process and $\mathbb{P}(\lim_{t \uparrow T} Z_t = b) = 1$, using that normally distributed random variables can converge in distribution if and only if the corresponding means and variances converge, we have $\lim_{t \uparrow T} \text{Var}(Z_t) = 0$, i.e.,

$$\lim_{t \uparrow T} \phi(t)^2 \int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du = 0,$$

and as a consequence, $\lim_{t \uparrow T} \phi(t) = 0$. Hence the result follows by Proposition 2.4. \square

3. EXAMPLES

3.1. Scaled Wiener bridges. Let $T \in (0, \infty)$ be fixed. For all $\alpha \in \mathbb{R}$, let us consider the SDE

$$(3.1) \quad \begin{cases} dZ_t = -\frac{\alpha}{T-t} Z_t dt + dB_t, & t \in [0, T), \\ Z_0 = 0, \end{cases}$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ satisfying the usual conditions. The SDE (3.1) has a pathwise unique strong solution, namely,

$$(3.2) \quad Z_t = \int_0^t \left(\frac{T-t}{T-s} \right)^\alpha dB_s, \quad t \in [0, T),$$

as it can be checked by Itô's formula. To our knowledge, these kinds of processes have been first considered in the case of $\alpha > 0$ by Brennan and Schwartz [6]; see also Mansuy [16]. Note also that in case of $\alpha = 1$ the process $(Z_t)_{t \in [0, T]}$ is nothing else but the usual Wiener bridge (from 0 to 0 over the time interval $[0, T]$).

It is known that in case of $\alpha > 0$, the process $(Z_t)_{t \in [0, T]}$ given by (3.2) has an almost surely continuous extension $(Z_t)_{t \in [0, T]}$ to the time-interval $[0, T]$ such that $Z_T = 0$ with probability one, see, e.g., Mansuy [16, page 1023] or Barczy and Pap [5, Lemma 3.1]. For positive values of α , the possibility of such an extension is based on a strong law of large numbers for square integrable local martingales. In case of $\alpha \leq 0$, there does not exist an almost surely continuous extension of the process $(Z_t)_{t \in [0, T]}$ to $[0, T]$ which would take some constant at time T with probability one (i.e., which would be a bridge). However, for all $\alpha \in \mathbb{R}$, the Gauss process $(Z_t)_{t \in [0, T]}$ given by (3.2) is called a scaled Wiener bridge or an α -Wiener bridge. More generally, we call any almost surely continuous (Gauss) process on the time interval $[0, T]$ having the same

finite-dimensional distributions as $(Z_t)_{t \in [0, T]}$ a scaled Wiener bridge (an α -Wiener bridge).

One can easily calculate

$$\text{Cov}(Z_s, Z_t) = \begin{cases} \frac{(T-s)^\alpha (T-t)^\alpha}{1-2\alpha} (T^{1-2\alpha} - (T-s \wedge t)^{1-2\alpha}) & \text{if } \alpha \neq \frac{1}{2}, \\ \sqrt{(T-s)(T-t)} (\ln(T) - \ln(T-s \wedge t)) & \text{if } \alpha = \frac{1}{2}, \end{cases}$$

for $s, t \in [0, T]$, see Barczy and Pap [5, Lemma 2.1].

Let $\phi : [0, T) \rightarrow \mathbb{R}$, $\phi(t) := (1 - t/T)^\alpha$, $t \in [0, T)$, $\psi : [0, T) \rightarrow \mathbb{R}$, $\psi(t) := 0$, $t \in [0, T)$, and $\sigma(t) := 1$, $t \in [0, T)$. Then the SDE (2.1) is nothing else but the SDE of an α -Wiener bridge, see (3.1), and

$$(3.3) \quad \ln \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) = \begin{cases} \ln \left(\frac{T^{2\alpha}}{1-2\alpha} (T^{1-2\alpha} - (T-t)^{1-2\alpha}) \right) & \text{if } \alpha \neq \frac{1}{2}, \\ \ln \left(T \ln \left(\frac{T}{T-t} \right) \right) & \text{if } \alpha = \frac{1}{2}, \end{cases}$$

for $t \in (0, T)$. In case of $\alpha > 0$, Theorem 2.5 can be applied with $b := 0$ and with

$$\varepsilon := \begin{cases} \frac{1}{2} & \text{if } 0 < \alpha \leq \frac{1}{2}, \\ \frac{1}{2(2\alpha-1)} & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

Indeed, if $0 < \alpha < 1/2$ and $\varepsilon = 1/2$, then

$$\phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{1/2+\varepsilon} = \left(1 - \frac{t}{T} \right)^\alpha \frac{T^{2\alpha}}{1-2\alpha} (T^{1-2\alpha} - (T-t)^{1-2\alpha}) \rightarrow 0$$

as $t \downarrow 0$ or $t \uparrow T$. If $\alpha = 1/2$ and $\varepsilon = 1/2$, then

$$\phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{1/2+\varepsilon} = \sqrt{1 - \frac{t}{T}} T \ln \left(\frac{T}{T-t} \right) \rightarrow T \ln(1) = 0 \quad \text{as } t \downarrow 0,$$

and, by \mathcal{L} 'Hospital's rule,

$$\lim_{t \uparrow T} \phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{1/2+\varepsilon} = \lim_{t \uparrow T} \frac{T \ln \left(\frac{T}{T-t} \right)}{\left(1 - \frac{t}{T} \right)^{-1/2}} = \lim_{t \uparrow T} 2\sqrt{T(T-t)} = 0.$$

If $\alpha > 1/2$ and $\varepsilon = 1/(2(2\alpha-1))$, then

$$\phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{1/2+\varepsilon} = \left(\frac{T^{2\alpha}}{2\alpha-1} \right)^{\alpha/(2\alpha-1)} \frac{1}{T^\alpha} (1 - T^{1-2\alpha} (T-t)^{2\alpha-1}),$$

which tends to 0 as $t \downarrow 0$ and to $(2\alpha-1)^{-\alpha/(2\alpha-1)} T^{\alpha/(2\alpha-1)}$ as $t \uparrow T$.

Then, by Theorem 2.5, there exists a strictly stationary centered Ornstein-Uhlenbeck process $R = (R_t)_{t \in \mathbb{R}}$ with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$, such that

$$Z_t = \sqrt{\text{Var}(Z_t)} R \left(\ln \left(\frac{T^{2\alpha}}{1-2\alpha} (T^{1-2\alpha} - (T-t)^{1-2\alpha}) \right) \right), \quad \forall t \in [0, T] \quad \text{a.s.}$$

in case $\alpha \neq \frac{1}{2}$, $\alpha > 0$, and

$$Z_t = \sqrt{\text{Var}(Z_t)} R \left(\ln \left(T \ln \left(\frac{T}{T-t} \right) \right) \right), \quad \forall t \in [0, T] \quad \text{a.s.}$$

in case $\alpha = \frac{1}{2}$, where

$$\sqrt{\text{Var}(Z_t)} = \begin{cases} (T-t)^\alpha \sqrt{\frac{T^{1-2\alpha} - (T-t)^{1-2\alpha}}{1-2\alpha}} & \text{if } \alpha \neq \frac{1}{2}, \\ \sqrt{(T-t) \ln \left(\frac{T}{T-t} \right)} & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

Remark 3.1. Note that if $\alpha = 1$ and $T = 1$, then

$$\begin{aligned} \text{Var}(Z_t) &= \sqrt{t(1-t)}, \quad t \in [0, 1], \\ \ln \left(\frac{T^{2\alpha}}{1-2\alpha} (T^{1-2\alpha} - (T-t)^{1-2\alpha}) \right) &= \ln \left(\frac{t}{1-t} \right), \quad t \in (0, 1), \end{aligned}$$

and, by Theorem 2.5, we get back the representation of a usual Wiener bridge (from 0 to 0 on the time interval $[0, 1]$) via a space-time scaled stationary Ornstein-Uhlenbeck process recalled in the Introduction. \square

Next we point out that the results presented for the α -Wiener bridge given by the SDE (3.1) can be generalized to the so-called general α -Wiener bridges introduced by Barczy and Kern [3, Section 3]. Let $T \in (0, \infty)$ be fixed, and for a continuously differentiable function $\alpha : [0, T] \rightarrow \mathbb{R}$, let us consider the SDE

$$(3.4) \quad \begin{cases} dZ_t = -\frac{\alpha(t)}{T-t} Z_t dt + dB_t, & t \in [0, T), \\ Z_0 = 0. \end{cases}$$

This SDE has a pathwise unique strong solution given by

$$(3.5) \quad Z_t = \int_0^t \exp \left\{ - \int_s^t \frac{\alpha(u)}{T-u} du \right\} dB_s, \quad t \in [0, T),$$

see, e.g., Barczy and Kern [3, Proposition 3.1]. Further, if $\alpha(T) := \lim_{t \uparrow T} \alpha(t)$ exists and $\alpha(T) > 0$, then the process $(Z_t)_{t \in [0, T)}$ given by (3.5) has an almost surely continuous extension $(Z_t)_{t \in [0, T]}$ to the time-interval $[0, T]$ such that $Z_T = 0$ with probability one, see, Barczy and Kern [5, Theorem 3.3].

Let $\phi : [0, T) \rightarrow \mathbb{R}$, $\phi(t) := \exp \left\{ - \int_0^t \frac{\alpha(u)}{T-u} du \right\}$, $t \in [0, T)$, $\psi : [0, T) \rightarrow \mathbb{R}$, $\psi(t) := 0$, $t \in [0, T)$, and $\sigma(t) := 1$, $t \in [0, T)$. Then the SDE (2.1) is nothing else but the SDE of a general α -Wiener bridge given in (3.4), and, by Theorem 2.1, there

exists a strictly stationary centered Ornstein-Uhlenbeck process $R = (R_t)_{t \in \mathbb{R}}$ with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$, such that

$$Z_t = \phi(t) \sqrt{\int_0^t \frac{1}{\phi(u)^2} du} R \left(\ln \left(\int_0^t \frac{1}{\phi(u)^2} du \right) \right), \quad \forall t \in [0, T], \quad \text{a.s.},$$

where the right hand side of the above equality at $t = 0$ is understood as an almost sure limit as $t \downarrow 0$.

Next we check that if $\alpha(T) := \lim_{t \uparrow T} \alpha(t)$ exists and $\alpha(T) > 0$, then Theorem 2.5 can be applied with $b := 0$ and with

$$\varepsilon := \begin{cases} \frac{1+2(\delta_1-\delta_2)}{2(2\delta_2-1)} & \text{if } \alpha(T) \geq \frac{1}{2}, \\ \frac{1}{2} & \text{if } 0 < \alpha(T) < \frac{1}{2}, \end{cases}$$

where δ_1 and δ_2 are chosen such that $0 < \delta_1 < \alpha(T) < \delta_2 < \delta_1 + 1/2$. We need to check that the function

$$\begin{aligned} & \phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{\frac{1}{2}+\varepsilon} \\ &= \exp \left\{ - \int_0^t \frac{\alpha(u)}{T-u} du \right\} \left(\int_0^t \exp \left\{ 2 \int_0^u \frac{\alpha(v)}{T-v} dv \right\} du \right)^{\frac{1}{2}+\varepsilon}, \quad t \in [0, T], \end{aligned}$$

is bounded. Let us choose a $t_0 \in (0, T)$ such that $\delta_1 \leq \alpha(t) \leq \delta_2$ for all $t \in [t_0, T]$.

First we consider the case $\alpha(T) \geq 1/2$. Since

$$\int_0^t \exp \left\{ 2 \int_0^u \frac{\alpha(v)}{T-v} dv \right\} du = C_1 + C_2 \int_{t_0}^t \exp \left\{ 2 \int_{t_0}^u \frac{\alpha(v)}{T-v} dv \right\} du, \quad t \in [t_0, T],$$

where

$$C_1 := \int_0^{t_0} \exp \left\{ 2 \int_0^u \frac{\alpha(v)}{T-v} dv \right\} du \quad \text{and} \quad C_2 := \exp \left\{ 2 \int_0^{t_0} \frac{\alpha(v)}{T-v} dv \right\},$$

we have for all $t \in [t_0, T]$

$$\begin{aligned} & \exp \left\{ - \int_0^t \frac{\alpha(u)}{T-u} du \right\} \left(\int_0^t \exp \left\{ 2 \int_0^u \frac{\alpha(v)}{T-v} dv \right\} du \right)^{\frac{1}{2}+\varepsilon} \\ & \leq C_3 \exp \left\{ -\delta_1 \int_{t_0}^t \frac{1}{T-v} dv \right\} \left(C_1 + C_2 \int_{t_0}^t \exp \left\{ 2\delta_2 \int_{t_0}^u \frac{1}{T-v} dv \right\} du \right)^{\frac{1}{2}+\varepsilon} \\ & = C_3 \left(\frac{T-t}{T-t_0} \right)^{\delta_1} \left(C_1 + C_2 \frac{(T-t_0)^{2\delta_2}}{2\delta_2-1} ((T-t)^{1-2\delta_2} - (T-t_0)^{1-2\delta_2}) \right)^{\frac{1}{2}+\varepsilon} \end{aligned}$$

$$\begin{aligned}
&\leq C_3 \left(\frac{T-t}{T-t_0} \right)^{\delta_1} \left(C_1 + C_2 \frac{(T-t_0)^{2\delta_2}}{2\delta_2-1} (T-t)^{1-2\delta_2} \right)^{\frac{1}{2}+\varepsilon} \\
&= C_3 \left(C_1 \left(\frac{T-t}{T-t_0} \right)^{\frac{2\delta_1}{2\varepsilon+1}} + \frac{C_2}{2\delta_2-1} (T-t_0)^{2\delta_2-\frac{2\delta_1}{2\varepsilon+1}} (T-t)^{1-2\delta_2+\frac{2\delta_1}{2\varepsilon+1}} \right)^{\frac{1}{2}+\varepsilon},
\end{aligned}$$

where

$$C_3 := \exp \left\{ - \int_0^{t_0} \frac{\alpha(u)}{T-u} du \right\}.$$

Here, using that $2\delta_2 - 1 > 0$, $\delta_2 - \delta_1 < 1/2$, and the explicit form of ε , one can easily verify that $1 - 2\delta_2 + \frac{2\delta_1}{2\varepsilon+1} > 0$, yielding that the function

$$\phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{\frac{1}{2}+\varepsilon}, \quad t \in [0, T],$$

is bounded in case of $\alpha(T) \geq 1/2$.

Next, we consider the case $0 < \alpha(T) < 1/2$. Additionally to $0 < \delta_1 < \alpha(T) < \delta_2 < \delta_1 + 1/2$, we can also assume that $\delta_2 < 1/2$. By the calculations for the case $\alpha(T) \geq 1/2$, we get

$$\begin{aligned}
\int_0^t \exp \left\{ 2 \int_0^u \frac{\alpha(v)}{T-v} dv \right\} du &= C_1 + C_2 \frac{(T-t_0)^{2\delta_2}}{2\delta_2-1} ((T-t)^{1-2\delta_2} - (T-t_0)^{1-2\delta_2}) \\
&\rightarrow C_1 + C_2 \frac{T-t_0}{1-2\delta_2} \quad \text{as } t \uparrow T,
\end{aligned}$$

and

$$\exp \left\{ - \int_0^t \frac{\alpha(v)}{T-v} dv \right\} \leq C_3 \left(\frac{T-t}{T-t_0} \right)^{\delta_1} \rightarrow 0 \quad \text{as } t \uparrow T,$$

where we used $1 - 2\delta_2 > 0$ and $\delta_1 > 0$. This yields that the function

$$\phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{\frac{1}{2}+\varepsilon}, \quad t \in [0, T],$$

is bounded also in case of $0 < \alpha(T) < 1/2$.

Concluding, if $\alpha(T) = \lim_{t \uparrow T} \alpha(t)$ exists and $\alpha(T) > 0$, then, by Theorem 2.5, there exists a strictly stationary centered Ornstein-Uhlenbeck process $R = (R_t)_{t \in \mathbb{R}}$ with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$, such that

$$Z_t = \phi(t) \sqrt{\int_0^t \frac{1}{\phi(u)^2} du} R \left(\ln \left(\int_0^t \frac{1}{\phi(u)^2} du \right) \right), \quad \forall t \in [0, T], \quad \text{a.s.},$$

where the right hand side of the above equation for $t = 0$ and for $t = T$ is understood as an almost sure limit as $t \downarrow 0$ and $t \uparrow T$, respectively.

3.2. Ornstein-Uhlenbeck type bridges. First we recall the notion and properties of Ornstein-Uhlenbeck type bridges to the extent needed. For a more detailed discussion and for the proofs of the results, see for example Barczy and Kern [4] (where one can also find extensions to multidimensional bridges).

Let us consider an Ornstein-Uhlenbeck type process $(X_t)_{t \in \mathbb{R}_+}$ given by the SDE

$$(3.6) \quad dX_t = q(t) X_t dt + \sigma(t) dB_t, \quad t \in \mathbb{R}_+,$$

with an initial condition X_0 having a Gauss distribution independent of B , where $q : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}$ are continuous functions and $(B_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ satisfying the usual conditions. By Itô's formula, there exists a pathwise unique strong solution of the SDE (3.6), namely, for $t \in \mathbb{R}_+$,

$$(3.7) \quad X_t = e^{\bar{q}(t)} \left(X_0 + \int_0^t e^{-\bar{q}(s)} \sigma(s) dB_s \right) \quad \text{with} \quad \bar{q}(t) := \int_0^t q(u) du.$$

Note that we may define the filtration $(\mathcal{A}_t)_{t \in \mathbb{R}_+}$ such that $\sigma\{X_0, B_s : 0 \leq s \leq t\} \subset \mathcal{A}_t$ for all $t \in \mathbb{R}_+$.

We call the process $(X_t)_{t \in \mathbb{R}_+}$ a Ornstein-Uhlenbeck process with continuously varying parameters, or a Gauss-Markov process of Ornstein-Uhlenbeck type with parameters $e^{\bar{q}}$ and σ .

Let us introduce the following notations and assumptions. Let

$$\gamma(s, t) := \int_s^t e^{2(\bar{q}(t) - \bar{q}(u))} \sigma^2(u) du < \infty, \quad 0 \leq s \leq t.$$

In what follows we will make the general assumption that

$$(3.8) \quad \sigma(t) \neq 0 \quad \text{for all } t \in \mathbb{R}_+.$$

This guarantees that $\gamma(s, t)$ is positive for all $0 \leq s < t$. Further, for all $a, b \in \mathbb{R}$ and $0 \leq s \leq t < T < \infty$, let

$$(3.9) \quad n_{a,b}(s, t) := \frac{\gamma(s, t)}{\gamma(s, T)} e^{\bar{q}(T) - \bar{q}(t)} b + \frac{\gamma(t, T)}{\gamma(s, T)} e^{\bar{q}(t) - \bar{q}(s)} a,$$

and

$$(3.10) \quad \sigma(s, t) := \frac{\gamma(s, t) \gamma(t, T)}{\gamma(s, T)}.$$

In Barczy and Kern [4], for fixed $T \in (0, \infty)$ and $a, b \in \mathbb{R}$ we constructed a Markov process $(Z_t)_{t \in [0, T]}$ with initial distribution $\mathbb{P}(Z_0 = a) = 1$ and with transition densities

$$(3.11) \quad p_{s,t}^Z(x, y) = \frac{p_{s,t}^X(x, y) p_{t,T}^X(y, b)}{p_{s,T}^X(x, b)}, \quad x, y \in \mathbb{R}, \quad 0 \leq s < t < T,$$

such that $Z_t \rightarrow b = Z_T$ almost surely and also in L^2 as $t \uparrow T$, where $p_{s,t}^X$ denotes the transition densities of X . The process $(Z_t)_{t \in [0, T]}$ is called a bridge of Ornstein-Uhlenbeck type from a to b over $[0, T]$ derived from X , see also Definition 3.3. The construction is based on Theorem 3.1 in Barczy and Kern [4], which we recall now for completeness and for our later purposes. For the proofs, see Barczy and Kern [4].

Theorem 3.2. *Let us suppose that condition (3.8) holds. For fixed $a, b \in \mathbb{R}$ and $T \in (0, \infty)$, let the process $(Z_t)_{t \in [0, T]}$ be given by*

$$(3.12) \quad Z_t = n_{a,b}(0, t) + \int_0^t \frac{\gamma(t, T)}{\gamma(s, T)} e^{\bar{q}(t) - \bar{q}(s)} \sigma(s) dB_s, \quad t \in [0, T].$$

Then for any $t \in [0, T)$ the distribution of Z_t is Gauss with mean $n_{a,b}(0, t)$ and with variance $\sigma(0, t)$. Especially, $Z_t \rightarrow b$ almost surely (and hence in probability) and in L^2 as $t \uparrow T$. Hence the process $(Z_t)_{t \in [0, T]}$ can be extended to an almost surely (and hence stochastically) and L^2 -continuous process $(Z_t)_{t \in [0, T]}$ with $Z_0 = a$ and $Z_T = b$. Moreover, $(Z_t)_{t \in [0, T]}$ is a Gauss-Markov process and for any $x \in \mathbb{R}$ and $0 \leq s < t < T$ the transition density $\mathbb{R} \ni y \mapsto p_{s,t}^Z(x, y)$ of Z_t given $Z_s = x$ is given by

$$p_{s,t}^Z(x, y) = \frac{1}{\sqrt{2\pi\sigma(s, t)}} \exp \left\{ -\frac{(y - n_{x,b}(s, t))^2}{2\sigma(s, t)} \right\}, \quad y \in \mathbb{R}.$$

Definition 3.3. Let $(X_t)_{t \in \mathbb{R}_+}$ be the process given by the SDE (3.6) with an initial Gauss random variable X_0 independent of $(B_t)_{t \in \mathbb{R}_+}$ and let us assume that condition (3.8) holds. For fixed $a, b \in \mathbb{R}$ and $T \in (0, \infty)$, the process $(Z_t)_{t \in [0, T]}$ defined in Theorem 3.2 is called a bridge of Ornstein-Uhlenbeck type from a to b over $[0, T]$ derived from X . More generally, we call any almost surely continuous (Gauss) process on the time-interval $[0, T]$ having the same finite-dimensional distributions as $(Z_t)_{t \in [0, T]}$ a bridge of Ornstein-Uhlenbeck type from a to b over $[0, T]$ derived from X .

One can also derive a SDE which is satisfied by the Ornstein-Uhlenbeck type bridge, see for example Theorem 3.3 in Barczy and Kern [4].

Theorem 3.4. *Let us suppose that condition (3.8) holds. The process $(Z_t)_{t \in [0, T]}$ defined by (3.12) is a pathwise unique strong solution of the linear SDE*

$$(3.13) \quad dZ_t = \left[\left(q(t) - \frac{e^{2(\bar{q}(T) - \bar{q}(t))}}{\gamma(t, T)} \sigma^2(t) \right) Z_t + \frac{e^{\bar{q}(T) - \bar{q}(t)}}{\gamma(t, T)} \sigma^2(t) b \right] dt + \sigma(t) dB_t$$

for $t \in [0, T]$ and with initial condition $Z_0 = a$.

By Lemma 2.7 in Barczy and Kern [4], one can easily calculate

$$\text{Cov}(Z_s, Z_t) = e^{\bar{q}(t) - \bar{q}(s)} \frac{\gamma(0, s) \gamma(t, T)}{\gamma(0, T)}, \quad 0 \leq s \leq t < T.$$

Let us define the mean centered Ornstein-Uhlenbeck type bridge

$$\tilde{Z}_t := Z_t - \mathbb{E}(Z_t) = Z_t - n_{a,b}(0, t), \quad t \in [0, T].$$

Note that $\mathbb{P}(\tilde{Z}_0 = 0) = \mathbb{P}(\tilde{Z}_T = 0) = 1$.

With the notations of Section 2.1, let $\xi := a$, $\phi : [0, T] \rightarrow (0, \infty)$ and $\psi : [0, T] \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} \phi(t) &:= \frac{\gamma(t, T) e^{\bar{q}(t)}}{\gamma(0, T)}, \quad t \in [0, T], \\ \psi(t) &:= \frac{e^{\bar{q}(T) - \bar{q}(t)}}{\gamma(t, T)} \sigma^2(t) b = \frac{e^{\bar{q}(T)}}{\gamma(0, T) \phi(t)} \sigma^2(t) b, \quad t \in [0, T]. \end{aligned}$$

Since

$$\frac{\phi'(t)}{\phi(t)} = q(t) - \sigma^2(t) \frac{e^{2(\bar{q}(T) - \bar{q}(t))}}{\gamma(t, T)}, \quad t \in [0, T],$$

with our special choices of ξ , ϕ and ψ , the SDE (2.1) is nothing else but the SDE of an Ornstein-Uhlenbeck bridge from a to b over time interval $[0, T]$, see (3.13). Further, using part (b) of Lemma A.3 in Barczy and Kern [4], one can check that

$$\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du = \frac{e^{-2\bar{q}(t)} \gamma(0, t) \gamma(0, T)}{\gamma(t, T)}, \quad t \in [0, T].$$

Theorem 2.5 can be applied with $b := 0$ and with $\varepsilon := 1/2$. Indeed,

$$\begin{aligned} \phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{1/2 + \varepsilon} &= \frac{\gamma(t, T) e^{\bar{q}(t)}}{\gamma(0, T)} \frac{e^{-2\bar{q}(t)} \gamma(0, t) \gamma(0, T)}{\gamma(t, T)} = e^{-\bar{q}(t)} \gamma(0, t) \\ &= e^{\bar{q}(t)} \int_0^t e^{-2\bar{q}(u)} \sigma(u)^2 du, \quad t \in [0, T], \end{aligned}$$

which is a bounded function, since the functions q and σ are continuous on \mathbb{R}_+ . Then, by Theorem 2.5, there exists a strictly stationary centered Ornstein-Uhlenbeck process $R = (R_t)_{t \in \mathbb{R}}$ with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$, such that

$$\tilde{Z}_t = \sqrt{\frac{\gamma(0, t)\gamma(t, T)}{\gamma(0, T)}} R \left(\ln \left(\frac{e^{-2\bar{q}(t)}\gamma(0, t)\gamma(0, T)}{\gamma(t, T)} \right) \right), \quad \forall t \in [0, T], \quad \text{a.s.},$$

where the right hand side of the above equation for $t = 0$ and for $t = T$ is understood as an almost sure limit as $t \downarrow 0$ and $t \uparrow T$, respectively.

Next we formulate the above presented results in the case of usual Ornstein-Uhlenbeck bridges.

Remark 3.5. In case of $q(t) = q \neq 0$, $t \in \mathbb{R}_+$, and $\sigma(t) = \sigma \neq 0$, $t \in \mathbb{R}_+$, the bridge of Ornstein-Uhlenbeck type $(Z_t)_{t \in [0, T]}$ from a to b over $[0, T]$ defined in (3.12) has the form

$$(3.14) \quad Z_t = a \frac{\sinh(q(T-t))}{\sinh(qT)} + b \frac{\sinh(qt)}{\sinh(qT)} + \sigma \int_0^t \frac{\sinh(q(T-s))}{\sinh(q(T-s))} dB_s$$

for $t \in [0, T)$ and $Z_T = b$, see, Remark 3.8 in Barczy and Kern [4]. In fact, the process $(Z_t)_{t \in [0, T]}$ is the pathwise unique strong solution of the SDE

$$dZ_t = q \left(-\coth(q(T-t))U_t + \frac{b}{\sinh(q(T-t))} \right) dt + \sigma dB_t, \quad t \in [0, T),$$

with an initial condition $Z_0 = a$, see, Remark 3.9 in Barczy and Kern [4]. Then, by Theorem 2.5, there exists a strictly stationary centered Ornstein-Uhlenbeck process $R = (R_t)_{t \in \mathbb{R}}$ with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$, such that

$$\tilde{Z}_t = \sqrt{\frac{\sigma^2 \sinh(qt) \sinh(q(T-t))}{q \sinh(qT)}} R \left(\ln \left(\frac{\sigma^2 \sinh(qt) \sinh(qT)}{q \sinh(q(T-t))} \right) \right), \quad \forall t \in [0, T], \quad \text{a.s.},$$

where the right hand side of the above equation for $t = 0$ and for $t = T$ is understood as an almost sure limit as $t \downarrow 0$ and $t \uparrow T$, respectively, since

$$\gamma(s, t) = \frac{\sigma^2}{q} e^{q(t-s)} \sinh(q(t-s)), \quad 0 \leq s \leq t,$$

and, by Barczy and Kern [4, formula (1.7)],

$$\text{Var } Z_t = \phi(t)^2 \int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du = \frac{\sigma^2 \sinh(qt) \sinh(q(T-t))}{q \sinh(qT)}, \quad t \in [0, T).$$

□

3.3. F -Wiener bridges. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a probability density function on \mathbb{R}_+ and let us consider the corresponding cumulative distribution function $F : \mathbb{R}_+ \rightarrow [0, 1]$, $F(t) := \int_0^t f(s) ds$, $t \in \mathbb{R}_+$. Further, let

$$T := \inf\{t \in \mathbb{R}_+ : F(t) = 1\} \in (0, \infty]$$

with the convention $\inf \emptyset := \infty$. Let us assume that f is continuous on $[0, T)$, and that there exists a $\delta \in (0, T)$ such that $f(t) \neq 0$ for all $t \in (0, \delta)$. We consider the SDE

$$(3.15) \quad dZ_t = -\frac{f(t)}{1-F(t)} Z_t dt + \sqrt{f(t)} dB_t, \quad t \in [0, T),$$

with an initial value $Z_0 = 0$, where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process. By Itô's formula,

$$Z_t = \int_0^t \frac{1-F(t)}{1-F(s)} \sqrt{f(s)} dB_s, \quad t \in [0, T),$$

is a pathwise unique strong solution of the SDE (3.15), and $(Z_t)_{t \in [0, T)}$ is a centered Gauss process with covariance function

$$\begin{aligned} \text{Cov}(Z_s, Z_t) &= (1-F(t))(1-F(s)) \int_0^{s \wedge t} \frac{f(u)}{(1-F(u))^2} du \\ &= (1-F(t))(1-F(s)) \frac{F(s \wedge t)}{1-F(s \wedge t)} = F(s \wedge t) - F(s)F(t) \end{aligned}$$

for $s, t \in [0, T)$. Note that $1-F(t)$, $t \in \mathbb{R}_+$, is nothing else but the survival function, and $\frac{f(t)}{1-F(t)}$, $t \in \mathbb{R}_+$, is the hazard rate (mean reversion rate) corresponding to the distribution function F . Let $\phi : [0, T) \rightarrow (0, \infty)$, $\phi(t) := 1-F(t)$, $t \in [0, T)$, $\psi : [0, T) \rightarrow \mathbb{R}$, $\psi(t) := 0$, $t \in [0, T)$, and $\sigma(t) := \sqrt{f(t)}$, $t \in [0, T)$. Then the SDE (2.1) is nothing else but the SDE (3.15), and

$$\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du = \int_0^t \frac{f(u)}{(1-F(u))^2} du = \int_0^t \frac{F'(u)}{(1-F(u))^2} du = \frac{F(t)}{1-F(t)}, \quad t \in [0, T).$$

By Theorem 2.1, there exists a strictly stationary centered Ornstein-Uhlenbeck process $R = (R_t)_{t \in \mathbb{R}}$ with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$, such that

$$Z_t = \sqrt{F(t)(1-F(t))} R \left(\ln \left(\frac{F(t)}{1-F(t)} \right) \right), \quad \forall t \in [0, T), \quad \text{a.s.},$$

where the right hand side of the above equation for $t = 0$ is understood as an almost sure limit as $t \downarrow 0$. Further, note that with $\varepsilon := 1/2$ we have

$$\phi(t) \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)^{1/2+\varepsilon} = (1 - F(t)) \frac{F(t)}{1 - F(t)} = F(t), \quad t \in [0, T),$$

which is a bounded function. Since F is a continuous distribution function, $\lim_{t \uparrow T} F(t) = 1$, and hence, by Proposition 2.4, we have $\mathbb{P}(\lim_{t \uparrow T} Z_t = 0) = 1$ and

$$Z_t = \sqrt{F(t)(1 - F(t))} R \left(\ln \left(\frac{F(t)}{1 - F(t)} \right) \right), \quad \forall t \in [0, T], \quad \text{a.s.},$$

where the right hand side of the above equation for $t = 0$ and for $t = T$ is understood as an almost sure limit as $t \downarrow 0$ and $t \uparrow T$, respectively. Then we can say that Z is a bridge over $[0, T]$ in the sense that its starting and ending points are zero (more precisely, $Z_0 = 0$ and $\mathbb{P}(\lim_{t \uparrow T} Z_t = 0) = 1$), and we can call Z as an F -Wiener bridge corresponding to the distribution function F . For more information on F -Wiener bridges (also called \mathbb{P} -Wiener bridges), see Shorack and Wellner [22, page 838] or van der Vaart [23, page 266].

To give an example, let us consider the cumulative distribution function $F : \mathbb{R}_+ \rightarrow [0, 1]$ defined by

$$F(t) := \begin{cases} 1 - (1 - \frac{t}{T})^\alpha & \text{if } t \in [0, T), \\ 1 & \text{if } t \geq T, \end{cases}$$

where $\alpha \in (0, \infty)$. Then $f(t) = \frac{\alpha}{T} (1 - \frac{t}{T})^{\alpha-1}$ for $t \in [0, T)$, and $f(t) = 0$ for $t \in \mathbb{R}_+ \setminus [0, T)$, $\inf\{t \in \mathbb{R}_+ : F(t) = 1\} = T$ and the SDE (3.15) of the F -Wiener bridges takes the form

$$dZ_t = -\frac{\alpha}{T-t} Z_t dt + \sqrt{f(t)} dB_t, \quad t \in [0, T),$$

with an initial value $Z_0 = 0$. Note that the drift coefficient of this SDE is the same as that of the SDE (3.1) of an α -Wiener bridge, however, the diffusion coefficients are different.

3.4. Weighted Wiener processes and weighted Wiener bridges. Let $(B_t)_{t \in \mathbb{R}_+}$ be a standard Wiener process, and $(B_t^\circ)_{t \in [0, 1]}$ be a Wiener bridge from 0 to 0 over $[0, 1]$. Let $w : \mathbb{R}_+ \rightarrow (0, \infty)$ be a continuously differentiable (weight) function such that $w(0) = 1$ (e.g., $w(t) = (1 + t)^\alpha$, $t \in \mathbb{R}_+$, with some $\alpha \in [1, \infty)$). Let us define

$$Z_t := w(t) B_t, \quad t \in \mathbb{R}_+, \quad \text{and} \quad Z_t^\circ := w(t) B_t^\circ, \quad t \in [0, 1].$$

The process $(Z_t)_{t \in \mathbb{R}_+}$ can be called a weighted Wiener process, and the process $(Z_t^\circ)_{t \in [0,1]}$ a weighted Wiener bridge. We note that Deheuvels and Martynov [8] considered weighted Wiener processes and weighted Wiener bridges with a weight function t^α for some $\alpha \in (0, \infty)$ (however, this weight function is not in our setup, since the condition $w(0) = 1$ does not hold). By (1.1) and Subsection 3.1, there exists a strictly stationary centered Ornstein-Uhlenbeck process $R = (R_t)_{t \in \mathbb{R}}$ with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$, such that

$$(3.16) \quad Z_t = w(t)\sqrt{t} R(\ln(t)), \quad \forall t \in \mathbb{R}_+ \quad \text{a.s.},$$

and

$$(3.17) \quad Z_t^\circ = w(t)\sqrt{t(1-t)} R\left(\ln\left(\frac{t}{1-t}\right)\right), \quad \forall t \in [0, 1] \quad \text{a.s..}$$

We point out that weighted Wiener processes and weighted Wiener bridges fit into our general framework (see Section 2), so that one can apply Theorem 2.1 and get the representations (3.16) and (3.17), detailed as follows. Namely, by Itô's formula,

$$(3.18) \quad dZ_t = \frac{w'(t)}{w(t)} Z_t dt + w(t) dB_t, \quad t \in \mathbb{R}_+,$$

$$(3.19) \quad dZ_t^\circ = \left(\frac{w'(t)}{w(t)} - \frac{1}{1-t} \right) Z_t^\circ dt + w(t) dB_t, \quad t \in [0, 1).$$

The SDEs (3.18) and (3.19) have the form (2.1) by choosing $T := \infty$, $\phi(t) := w(t)$, $t \in \mathbb{R}_+$, $\psi(t) := 0$, $t \in \mathbb{R}_+$, $\sigma(t) := w(t)$, $t \in \mathbb{R}_+$, and $T := 1$, $\phi(t) := w(t)(1-t)$, $t \in [0, 1)$, $\psi(t) := 0$, $t \in [0, 1)$, $\sigma(t) := w(t)$, $t \in [0, 1)$, respectively. Concerning the time scalings, an easy calculation shows that for the SDE (3.18), we have

$$\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du = t, \quad t \in \mathbb{R}_+,$$

and for the SDE (3.19),

$$\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du = \frac{t}{1-t}, \quad t \in [0, 1),$$

as desired.

4. COUNTEREXAMPLES

In this section we give counterexamples for bridge processes that cannot be represented as a space-time scaled stationary Ornstein-Uhlenbeck process.

4.1. Zero area Wiener bridge. Let $(B_t^\circ)_{t \in [0,1]}$ be a Wiener bridge from 0 to 0 over $[0, 1]$, and let us consider the process

$$B_t^\circ - 6t(1-t) \int_0^1 B_u^\circ du, \quad t \in [0, 1],$$

introduced by Deheuvels [9]. According to page 1191 in Deheuvels [9], this process coincides in law with a zero area Wiener bridge $(M_t)_{t \in [0,1]}$, which is defined by conditioning a standard Wiener process $(B_t)_{t \in [0,1]}$ such that $B_1 = 0$ and $\int_0^1 B_u du = 0$ (for a precise definition, see G3rgens [11, Section 1.1]). The zero area Gauss process $(M_t)_{t \in [0,1]}$ has covariance function

$$\text{Cov}(M_s, M_t) = s \wedge t - st - 3st(1-s)(1-t), \quad s, t \in [0, 1],$$

see, Deheuvels [9, Lemma 2.1 with $K = 1$].

We check that one cannot find a monotone function $\tau : [0, 1] \rightarrow \mathbb{R}$ such that

$$(4.1) \quad \text{Cov}(M_s, M_t) = \sqrt{\text{Var}(M_s)} \sqrt{\text{Var}(M_t)} \text{Cov}(R_{\tau(s)}, R_{\tau(t)}), \quad s, t \in [0, 1],$$

where $R = (R_t)_{t \in \mathbb{R}}$ is a strictly stationary centered Ornstein-Uhlenbeck process with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$. On the contrary, let us suppose that there exists such a function τ . Then, due to the covariance structure of R , the covariance function $\text{Cov}(M_s, M_t)$, $s, t \in [0, 1]$, would be written as a product of a function only of s and a function only of t , i.e., there would exist some functions $f : [0, 1] \rightarrow \mathbb{R}$ and $g : [0, 1] \rightarrow \mathbb{R}$ such that $\text{Cov}(M_s, M_t) = f(s)g(t)$, $s, t \in [0, 1]$. Then for all $0 \leq s \leq t \leq 1$, we have

$$\text{Cov}(M_s, M_t) = s - st - 3st(1-s)(1-t) = s(1-t)(1-3t(1-s)) = f(s)g(t),$$

which yields that

$$(4.2) \quad 1 - 3t(1-s) = \frac{f(s)}{s} \frac{g(t)}{1-t} =: \tilde{f}(s)\tilde{g}(t), \quad 0 < s \leq t < 1.$$

By substituting $s := 1/2$ into (4.2), we have

$$1 - \frac{3}{2}t = \tilde{f}(1/2)\tilde{g}(t), \quad t \in [1/2, 1),$$

and hence $\tilde{f}(1/2) \neq 0$ and

$$\tilde{g}(t) = \frac{1 - \frac{3}{2}t}{\tilde{f}(1/2)}, \quad t \in [1/2, 1).$$

By substituting $t := 1/2$ into (4.2),

$$1 - \frac{3}{2}(1 - s) = \tilde{f}(s)\tilde{g}(1/2) = \tilde{f}(s)\frac{1}{4\tilde{f}(1/2)}, \quad s \in (0, 1/2].$$

Then

$$\tilde{f}(s) = 4\tilde{f}(1/2) \left(1 - \frac{3}{2}(1 - s)\right), \quad s \in (0, 1/2].$$

Hence,

$$\tilde{f}(s)\tilde{g}(t) = 4 \left(1 - \frac{3}{2}(1 - s)\right) \left(1 - \frac{3}{2}t\right) = -2 + 6s + 3t - 9st$$

for $s \in (0, 1/2]$ and $t \in [1/2, 1)$. Using (4.2), by choosing, e.g., $s := 1/4$ and $t := 2/3$, we arrive at a contradiction, since $1 - 3t(1 - s) = -1/2$ and $-2 + 6s + 3t - 9st = 0$. Hence the law of $(M_t)_{t \in [0,1]}$ cannot be the same as the law of $(\sqrt{\text{Var}(M_t)} R_{\tau(t)})_{t \in [0,1]}$ for any monotone function $\tau : [0, 1] \rightarrow \mathbb{R}$.

Next, we present another short proof for the fact that one cannot find a monotone function $\tau : [0, 1] \rightarrow \mathbb{R}$ such that (4.1) holds, communicated to us by Yakov Nikitin. The right hand side of (4.1) is non-negative for all $s, t \in [0, 1]$, but one can choose a sufficiently small $\varepsilon \in (0, 1)$ such that $\text{Cov}(M_\varepsilon, M_{1-\varepsilon})$ is negative. Indeed,

$$\text{Cov}(M_\varepsilon, M_{1-\varepsilon}) = \varepsilon - \varepsilon(1 - \varepsilon) - 3\varepsilon^2(1 - \varepsilon)^2 = \varepsilon^2(-3\varepsilon^2 + 6\varepsilon - 2),$$

which is negative for sufficiently small $\varepsilon \in (0, 1)$.

We note that, for a zero area Wiener bridge $(M_t)_{t \in [0,1]}$, Deheuvels [9, Theorem 1.2] derived a Karhunen–Loève expansion, Nazarov [17, Theorem 1] investigated small ball probabilities, and Görgens [11, Section 6.1] presented a SDE with $(M_t)_{t \in [0,1]}$ as a strong solution. However, up to our knowledge, no integral representation is available for $(M_t)_{t \in [0,1]}$, so $(M_t)_{t \in [0,1]}$ does not fit into our framework or into the one of Lachout [14].

4.2. Glued Wiener bridge. We present another simple counterexample initiated by Helmut Finner. Namely, if we take two independent Wiener bridges from 0 to 0 over $[0, 1]$ and over $[1, 2]$, respectively, and glue them together, then it is a Gauss bridge from 0 to 0 over $[0, 2]$, but it cannot be represented as a space-time transformed stationary Ornstein–Uhlenbeck process. Indeed, if $(B_t^\circ)_{t \in [0,1]}$ and $(W_t^\circ)_{t \in [0,1]}$ are independent Wiener bridges from 0 to 0 over $[0, 1]$, then for the so-called glued

Wiener bridge $G_t := B_t^\circ \mathbf{1}_{[0,1]}(t) + W_{t-1}^\circ \mathbf{1}_{[1,2]}(t)$, $t \in [0, 2]$, one cannot find a monotone function $\tau : [0, 2] \rightarrow \mathbb{R}$ such that

$$\text{Cov}(G_s, G_t) = \sqrt{\text{Var}(G_s)} \sqrt{\text{Var}(G_t)} \text{Cov}(R_{\tau(s)}, R_{\tau(t)}) \quad \text{for all } s, t \in [0, 2],$$

where $R = (R_t)_{t \in \mathbb{R}}$ is a strictly stationary centered Ornstein-Uhlenbeck process with $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$. On the contrary, let us suppose that there exists such a function τ . Then for $s = \frac{1}{2}$ and $t = \frac{3}{2}$, we would have $\text{Cov}(G_{1/2}, G_{3/2}) = 0$, and

$$\sqrt{\text{Var}(G_{1/2})} \sqrt{\text{Var}(G_{3/2})} \text{Cov}(R_{\tau(1/2)}, R_{\tau(3/2)}) = \frac{1}{4} e^{-\frac{|\tau(3/2) - \tau(1/2)|}{2}} > 0,$$

leading to a contradiction.

5. AN APPLICATION

First, we give a reformulation of Theorem 2.1. If we standardize the process Z , then it can be represented as

$$Z_t^* := \frac{Z_t - \mathbb{E}(Z_t)}{\sqrt{\text{Var}(Z_t)}} = R \left(\ln \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right) \right), \quad \forall t \in (0, T) \quad \text{a.s.},$$

for some strictly stationary centered Ornstein-Uhlenbeck process $R = (R_t)_{t \in \mathbb{R}}$ such that $\text{Cov}(R_s, R_t) = e^{-\frac{|t-s|}{2}}$, $s, t \in \mathbb{R}$.

This enables us to handle the distribution of the supremum location of $(Z_t^*)_{t \in (0, T)}$ on compact subintervals of $(0, T)$. Namely, for a stochastic process $X = (X_t)_{t \in \mathbb{R}}$ with continuous sample paths and for an interval $[a, b]$, $a < b$, $a, b \in \mathbb{R}$, let

$$\tau_{X,[a,b]} := \inf \left\{ t \in [a, b] : X_t = \sup_{s \in [a,b]} X_s \right\} \quad \text{and} \quad M_{X,[a,b]} := \sup_{s \in [a,b]} X_s,$$

i.e., $\tau_{X,[a,b]}$ is the leftmost time the overall supremum $M_{X,[a,b]}$ in the interval $[a, b]$ is achieved. We mention that the almost sure uniqueness of the supremum location on compact intervals of \mathbb{R}_+ for a continuous Gauss process X satisfying $\text{Var}(X_t - X_s) \neq 0$ for $s \neq t$, $s, t \in \mathbb{R}_+$, was proved in Kim and Pollard [13, Lemma 2.6]. Recently, Pimental [19, Theorem 1] has given necessary and sufficient conditions for the almost sure uniqueness of the supremum location on compact intervals for stochastic processes with continuous trajectories. If the set $\{t \in [0, T) : \sigma(t) = 0\}$ does not contain any interval and $\lim_{t \uparrow T} \int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du =: S \in (0, \infty]$, then the continuous function $\beta : (0, T) \rightarrow (-\infty, \ln(S))$, $\beta(t) := \ln \left(\int_0^t \frac{\sigma(u)^2}{\phi(u)^2} du \right)$, $t \in (0, T)$, is strictly

increasing having inverse $\beta^{-1} : (-\infty, \ln(S)) \rightarrow (0, T)$. Consequently, using that R is strictly stationary, we have

$$\tau_{Z^*, [t_1, t_2]} = \beta^{-1}(\tau_{R, [\beta(t_1), \beta(t_2)]}) \stackrel{\mathcal{D}}{=} \beta^{-1}(\beta(t_1) + \tau_{R, [0, \beta(t_2) - \beta(t_1)]})$$

for any $0 < t_1 < t_2 < T$, where the first equality holds almost surely and $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution. So we reduced the problem to handle the distribution of the supremum location of R on compact subintervals of the form $[0, T]$, $T \in \mathbb{R}_+$.

Samorodnitsky and Shen [21] provide a lot of information on the distribution of $\tau_{R, [0, T]}$, $T > 0$. By Theorem 3.1 in Samorodnitsky and Shen [21], the restriction of the law of $\tau_{R, [0, T]}$ to the interior $(0, T)$ of the interval $[0, T]$ is absolutely continuous and very specific properties of the density function in question have been described, e.g., the (càdlàg version of) the density function at t has a universal upper bound given by $\max(\frac{1}{t}, \frac{1}{T-t})$, $t \in (0, T)$. Using that R is time reversible, i.e., the laws of the processes $(R_{-t})_{t \in \mathbb{R}}$ and $(R_t)_{t \in \mathbb{R}}$ coincide, a finer upper bound for the density function in question has been derived, see Samorodnitsky and Shen [21, Proposition 4.2]. Further, this density function is not bounded near each of the two endpoints of the interval $[0, T]$, and $\mathbb{P}(\tau_{R, [0, T]} = 0) = \mathbb{P}(\tau_{R, [0, T]} = T) = 0$, $T > 0$, see Samorodnitsky and Shen [21, Example 3.7]. Later it will turn out that the law of $\tau_{R, [0, T]}$ (without restriction to $(0, T)$) is absolutely continuous, and we will derive an expression for its density function as well. We point out that, compared to the general setup of Samorodnitsky and Shen [21], we can take the advantage that R is not only strictly stationary, but a time-homogeneous Markov process as well, and hence we can use some general result of Csáki et al. [7] to handle the distribution of the supremum location of R .

In what follows we present a procedure which results in a (hopefully) numerically tractable formula for the density function of the distribution of the supremum location of R on a compact interval of the form $[0, T]$, $T \in \mathbb{R}_+$. First, recall that the law of $(R_t)_{t \in \mathbb{R}_+}$ can be represented as the law of the pathwise unique strong solution of an appropriate SDE. Namely, if $(B_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process and ξ is a standard normally distributed random variable independent of $(B_t)_{t \in \mathbb{R}_+}$, then the process

$$V_t := e^{-\frac{t}{2}} \left(\xi + \int_0^t e^{\frac{r}{2}} dB_r \right), \quad t \in \mathbb{R}_+,$$

is the pathwise unique strong solution of the SDE

$$dV_t = -\frac{1}{2}V_t dt + dB_t, \quad t \in \mathbb{R}_+,$$

with initial condition $V_0 = \xi$, and $(V_t)_{t \in \mathbb{R}_+}$ generates the same measure on $C(\mathbb{R}_+)$ as $(R_t)_{t \in \mathbb{R}_+}$. The mapping

$$C([0, T]) \ni f \mapsto (M_{f,[0,T]}, f(T), \tau_{f,[0,T]}) \in \mathbb{R} \times \mathbb{R} \times [0, T]$$

is measurable for all $T > 0$, since $\{\tau_{f,[0,T]} \leq t\} = \{M_{f,[0,t]} \geq M_{f,[t,T]}\}$ for all $t \in [0, T]$. Hence the laws of $(M_{V,[0,T]}, V_T, \tau_{V,[0,T]})$ and $(M_{R,[0,T]}, R_T, \tau_{R,[0,T]})$ coincide for all $T > 0$. Using that the so-called scale function and speed measure (see, e.g., Karatzas and Shreve [12, Section 5, formulae (5.42) and (5.51)]) corresponding to V take the forms

$$\begin{aligned} S_c(x) &= \int_c^x \exp \left\{ -2 \int_c^y -\frac{z}{2} dz \right\} dy = \int_c^x e^{\frac{y^2 - c^2}{2}} dy, \quad x \in \mathbb{R}, \quad c \in \mathbb{R}, \\ m_c(B) &= \int_B 2e^{-\frac{x^2 - c^2}{2}} dx, \quad B \in \mathcal{B}(\mathbb{R}), \quad c \in \mathbb{R}, \end{aligned}$$

by Theorem A in Csáki et al. [7], we get

$$\begin{aligned} &\mathbb{P}(M_{R,[0,T]} \in A, R_T \in B, \tau_{R,[0,T]} \in C \mid R_0 = x) \\ &= \int_A \int_B \int_C n_x(s, y) n_z(T - s, y) \mathbf{1}_{\{x \leq y\}} \mathbf{1}_{\{z \leq y\}} S_c(dy) m_c(dz) ds \\ &= \int_A \int_B \int_C n_x(s, y) n_z(T - s, y) 2e^{-\frac{z^2 - c^2}{2}} e^{\frac{y^2 - c^2}{2}} \mathbf{1}_{\{x \leq y\}} \mathbf{1}_{\{z \leq y\}} dy dz ds \\ &= \int_A \int_B \int_C n_x(s, y) n_z(T - s, y) 2e^{-\frac{z^2 - y^2}{2}} \mathbf{1}_{\{x \leq y\}} \mathbf{1}_{\{z \leq y\}} dy dz ds \end{aligned}$$

for all $x \in \mathbb{R}$, $A, B \in \mathcal{B}(\mathbb{R})$ and $C \in \mathcal{B}([0, T])$, where $(0, \infty) \ni u \mapsto n_x(u, y)$ denotes the (conditional) density function of the random variable $\inf\{t \in (0, \infty) : R_t = y\}$ provided that $R_0 = x$, where $x, y \in \mathbb{R}$. Note that the above formula does not depend on $c \in \mathbb{R}$. In Alili et al. [1] one can find several formulae for $n_x(u, y)$, $u \in (0, \infty)$, e.g., due to their formula (4.1), for all $x < y$, $x, y \in \mathbb{R}$, and $u \in (0, \infty)$,

$$n_x(u, y) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{u\alpha}{2}\right) \frac{Hr_{-\alpha}\left(\frac{-y}{\sqrt{2}}\right) Hr_{-\alpha}\left(\frac{-x}{\sqrt{2}}\right) + Hi_{-\alpha}\left(\frac{-x}{\sqrt{2}}\right) Hi_{-\alpha}\left(\frac{-y}{\sqrt{2}}\right)}{\left(Hr_{-\alpha}\left(\frac{-y}{\sqrt{2}}\right)\right)^2 + \left(Hi_{-\alpha}\left(\frac{-y}{\sqrt{2}}\right)\right)^2} d\alpha,$$

where, for all $\alpha \in \mathbb{R}$, the functions

$$Hr_\alpha(v) := \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} \cos\left(\frac{\alpha}{2} \log\left(1 + \left(\frac{v}{s}\right)^2\right)\right) ds, \quad v \in \mathbb{R},$$

$$Hi_\alpha(v) := \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} \sin\left(\frac{\alpha}{2} \log\left(1 + \left(\frac{v}{s}\right)^2\right)\right) ds, \quad v \in \mathbb{R},$$

are the real and imaginary parts of certain normalized Hermite functions, respectively. Hence, by the law of total probability and Fubini's theorem,

$$\begin{aligned} & \mathbb{P}(\tau_{R,[0,T]} \in C) \\ &= \int_{-\infty}^\infty \mathbb{P}(M_{R,[0,T]} \in \mathbb{R}, R_T \in \mathbb{R}, \tau_{R,[0,T]} \in C \mid R_0 = x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \int_C \left(\int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty n_x(s, y) n_z(T-s, y) 2e^{-\frac{z^2-y^2}{2}} \mathbf{1}_{\{x \leq y\}} \mathbf{1}_{\{z \leq y\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx dy dz \right) ds \end{aligned}$$

for all $C \in \mathcal{B}([0, T])$. Consequently, the density function of $\tau_{R,[0,T]}$ can be chosen as

$$f_{\tau_{R,[0,T]}}(s) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty \left(\int_x^\infty \left(\int_{-\infty}^y n_x(s, y) n_z(T-s, y) e^{-\frac{x^2+z^2-y^2}{2}} dz \right) dy \right) dx$$

for $s \in (0, T)$. Note that, by Samorodnitsky and Shen [21, Theorems 3.1, 3.3 and Example 3.7], $f_{\tau_{R,[0,T]}}$ is continuous on $(0, T)$, $\lim_{s \downarrow 0} f_{\tau_{R,[0,T]}}(s) = \infty$ and $\lim_{s \uparrow T} f_{\tau_{R,[0,T]}}(s) = \infty$.

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